

# Scaling Limits of Solutions of SPDE Driven by Lévy White Noises

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## Abstract

Consider a random process  $s$  solution of the stochastic partial differential equation  $Ls = w$  with  $L$  a homogeneous operator and  $w$  a multidimensional Lévy white noise. In this paper, we study the asymptotic effect of a zoom or a de-zoom on the process  $s$ . More precisely, we give sufficient conditions on  $L$  and  $w$  so that  $a^H s(\cdot/a)$  converges in law to a self-similar process for some  $H$ , when  $a \rightarrow 0$  (coarse-scale behavior) and  $a \rightarrow \infty$  (fine-scale behavior). The parameter  $H$  depends on the homogeneity order of the operator  $L$  and the Blumenthal-Gettoor indices associated to the Lévy white noise  $w$ . Finally, we apply our general results to several notorious classes of random processes and random fields.

**Keywords.** Lévy white noises, stochastic PDE, scaling limit, self-similar processes

**AMS subject classifications.** 60H15, 60F05, 60G18, 60G20

## 1 Introduction

A Lévy process is a stochastically continuous random process  $X = (X(t))$  with stationary and independent increments. When the laws of  $X(t)$  are symmetric- $\alpha$ -stable (S $\alpha$ S), the process  $X$  is self-similar, which means that it is invariant (in law) under a rescaling operation up to an adequate renormalization [36]. More precisely, for the S $\alpha$ S process  $X_\alpha$  with  $0 < \alpha \leq 2$ , we have that

$$a^{1/\alpha} X_\alpha(t/a) \stackrel{(d)}{=} X_\alpha(t) \quad (1)$$

for every  $t \in \mathbb{R}$  and  $a > 0$ . The case  $\alpha = 2$  corresponds to Brownian motion. However, if the noise is not stable, then the Lévy process  $X$  is no longer self-similar.

The study of self-similar processes and self-similar fields is a branch of probability theory [14]. Self-similar processes and fields have been applied in areas such as signal and image processing [4, 16, 34] or traffic network [26, 30], among others [28]. Many notorious random processes are self-similar, starting with fractional Brownian motions [29] and their

higher-order extensions [33]. It also allows for infinite-variance stable processes [36] and their fractional versions [21]. Self-similar random fields have also been investigated both in the Gaussian [2, 12, 27, 40] and the  $\alpha$ -stable case [1, 2].

Self-similar processes are intimately linked with stable laws [36]. Indeed, stable laws are the only possible probabilistic limits of the renormalized sum of independent and identically distributed random variables: This is the well-known (generalized) central-limit theorem [32, Theorem 1.20]. Due to this result, self-similar processes are often scaling limits of many discretization schemes and stochastic models [3, 8, 13, 24, 38].

In this paper, we focus on the impact of rescaling operations for a broader class of random processes that are asymptotically self-similar. These processes are specified as the solutions of a stochastic differential equation of the form

$$Ls = w, \tag{2}$$

where  $L$  is a differential operator on the functions from  $\mathbb{R}^d$  to  $\mathbb{R}$  and  $w$  a multidimensional Lévy white noise. Our aim is to study the statistical behavior of the rescaling  $\mathbf{x} \mapsto s(\mathbf{x}/a)$  of a solution of (2) when  $a > 0$  is varying. Our two main questions are:

- What is the asymptotic behavior of  $s(\cdot/a)$  when we zoom out the process (*i.e.*, when  $a \rightarrow 0$ )?
- What is the asymptotic behavior when we zoom in (*i.e.*, when  $a \rightarrow \infty$ )?

Our main contribution is to identify the sufficient conditions such that the rescaling  $a^H s(\cdot/a)$  of a solution of (2) has a self-similar asymptotic limit as  $a$  goes to 0 or  $\infty$ . When this limit exists, the parameter  $H$  is unique and depends essentially on the degree of homogeneity  $\gamma$  of  $L$  and on the Blumenthal-Gettoor indices  $\beta_0, \beta_\infty$  of  $w$  [5, 35]. We recall that the indices  $\beta_0$  and  $\beta_\infty$  are used in the literature to characterize the asymptotic and local behaviors of a Lévy process that is not self-similar [6].

The paper is organized as follows: In Section 2, we introduce the framework of generalized random processes, while the general class of random processes of interest is addressed in Section 3. We present our main results in Section 4, where we give sufficient conditions under which the asymptotic behavior of  $a^H s(a\cdot)$  is known at coarse and fine scales. Finally, we apply in Section 5 our result to specific classes of random processes, for different types of white noises and operators.

## 2 Generalized Random Processes

The theory of generalized random processes was introduced independently in the 50's by K. Itô [22] and I. Gelfand [19]. Among the benefits of this theory to the construction and study of random processes, we mention:

- *Its generality.* It allows one to define the broadest class of linear processes, including processes with no pointwise interpretation such as Lévy white noises.
- *The availability of an infinite-dimensional Bochner theorem.* The characteristic functional (see Definition 2) characterizes the law of a generalized random process in

the same way that the characteristic function does for random variables. The theory provides a complete description of the class of valid characteristic functionals based on higher-level properties (see Theorem 1 below). This allows for the construction of generalized random processes via their characteristic functional.

- *An infinite-dimensional Lévy continuity theorem.* The convergence in law of a sequence of random vectors is equivalent to the pointwise convergence of the corresponding characteristic functions. This result does not admit a direct generalization for processes defined over infinite-dimensional Hilbert spaces; however, it becomes true again for the generalized random processes defined over the extended space of tempered distribution  $\mathcal{S}'(\mathbb{R}^d)$  (see Theorem 2). This provides a powerful tool to show the convergence in law of random processes. We shall exploit this tool extensively in this paper.

## 2.1 Definition

The space of rapidly decaying and infinitely differentiable functions is denoted by  $\mathcal{S}(\mathbb{R}^d)$  and endowed with its usual Fréchet topology. Its continuous dual, the space of tempered distribution, is  $\mathcal{S}'(\mathbb{R}^d)$ . We fix a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ . The space of real-valued random variables  $L^0(\Omega)$  is endowed with the Fréchet topology associated with the convergence in probability. In particular, the sequence of random variables  $(X_n)$  vanishes in  $L^0(\Omega)$  if and only if  $\mathbb{E}[|X_n| \wedge 1] \xrightarrow{n \rightarrow \infty} 0$ .

**Definition 1.** A *generalized random process*  $s$  is a collection of random variables  $\langle s, \varphi \rangle \in L^0(\Omega)$  indexed by  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  that satisfy the two following properties:

- *Linearity.* For all  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$  and  $\lambda \in \mathbb{R}$ ,  $\langle s, \varphi_1 \rangle + \lambda \langle s, \varphi_2 \rangle = \langle s, \varphi_1 + \lambda \varphi_2 \rangle$  almost surely.
- *Continuity.* If  $\varphi_n \xrightarrow{n \rightarrow \infty} \varphi$  in  $\mathcal{S}(\mathbb{R}^d)$ , then  $\langle s, \varphi_n \rangle \xrightarrow{n \rightarrow \infty} \langle s, \varphi \rangle$  in  $L^0(\mathbb{R}^d)$ .

The generalized random process  $s$  in Definition 1 is specified as a random linear functional over the space  $\mathcal{S}(\mathbb{R}^d)$ . Alternatively, one can see  $s$  as a random variable with values in  $\mathcal{S}'(\mathbb{R}^d)$ , which would be a measurable map from  $\Omega$  to  $\mathcal{S}'(\mathbb{R}^d)$ , where  $\mathcal{S}'(\mathbb{R}^d)$  is endowed with its cylindrical  $\sigma$ -field. The equivalence between these two notions—random linear functional on  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$ -valued random variable—is not obvious and comes from the structure of the nuclear countable multi-Hilbert space of  $\mathcal{S}(\mathbb{R}^d)$ . It means in particular that it is admissible to look at  $s$  as a random tempered generalized function. More details on the different characterizations of generalized random processes can be found in [23].

## 2.2 Characteristic Functional

**Definition 2.** The *characteristic functional* of a generalized random process  $s$  is defined for  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  as

$$\widehat{\mathcal{P}}_s(\varphi) = \mathbb{E} \left[ e^{i\langle s, \varphi \rangle} \right]. \quad (3)$$

As announced in the Introduction of Section 2, we present the generalizations of the Bochner and Lévy continuity theorems for generalized random processes.

**Theorem 1.** *A functional  $\widehat{\mathcal{P}}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$  is the characteristic functional of a generalized random processes if and only if*

- *it is continuous over  $\mathcal{S}(\mathbb{R}^d)$ ;*
- *it is positive-definite, meaning that*

$$\sum_{n,m=1}^N a_n a_m^* \widehat{\mathcal{P}}(\varphi_n - \varphi_m) \geq 0 \quad (4)$$

*for any  $N \geq 1$ ,  $a_1, \dots, a_N \in \mathbb{C}$ ,  $\varphi_1, \dots, \varphi_N \in \mathcal{S}(\mathbb{R}^d)$ ; and*

- $\widehat{\mathcal{P}}(0) = 1$ .

This result is called the Minlos-Bochner theorem. It is true more generally for any functional defined over a nuclear space [31]. The nuclear structure of  $\mathcal{S}(\mathbb{R}^d)$  is at the heart of the theory of generalized random processes.

**Definition 3.** A sequence of generalized random processes  $(s_n)$  converges in law to the generalized process  $s$  if, for any  $\varphi_1, \dots, \varphi_N \in \mathcal{S}(\mathbb{R}^d)$ , the sequence of random vectors  $(\langle s_n, \varphi_1 \rangle, \dots, \langle s_n, \varphi_N \rangle)$  converges in law to the random vector  $(\langle s, \varphi_1 \rangle, \dots, \langle s, \varphi_N \rangle)$ .

**Theorem 2.** *A sequence of generalized random processes  $(s_n)$  converges in law to the generalized random process  $s$  if and only if*

$$\widehat{\mathcal{P}}_{s_n}(\varphi) \xrightarrow[n \rightarrow \infty]{} \widehat{\mathcal{P}}_s(\varphi) \quad (5)$$

*for any  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .*

A proof of this result for general nuclear spaces was obtained by P. Boulicaut [7].

## 2.3 Lévy White Noises and Infinite Divisibility

The construction of continuous-domain Lévy white noises and related processes is intimately linked with the infinite divisibility of the finite-dimensional marginals of those processes. The main idea is the following: If  $(s(t))_{t \geq 0}$  is a stochastically continuous point-wise process with stationary and independent increments and  $s(0) = 0$  (in other terms, if  $s$  is a Lévy process), then we can decompose for all  $N \geq 1$  and  $t \geq 0$  the random variable  $s(t)$  as

$$s(t) = \sum_{n=1}^N \left( s\left(n \frac{t}{N}\right) - s\left((n-1) \frac{t}{N}\right) \right) := \sum_{n=1}^N X_{n,N} \quad (6)$$

where the  $X_{n,N}$ ,  $n = 1, \dots, N$ , are independent (since the increments are independent) and identically distributed (since the increments are stationary). This is precisely the definition of an infinitely divisible random variable [37].

If  $X$  is infinitely divisible, then its characteristic function can be written as

$$\widehat{\mathcal{P}}_X(\xi) = \exp(\psi(\xi)) \quad (7)$$

with  $\psi$  a continuous function [37, Lemma 7.6].

**Definition 4.** The continuous log-characteristic function of an infinitely divisible random variable is called a *Lévy exponent*.

It can be shown that  $\psi$  is a valid Lévy exponent if and only if  $\exp(\tau\psi(\xi))$  is a valid characteristic function for every  $\tau > 0$ .

There is a one-to-one correspondence between infinitely divisible random variables and Lévy processes, so that the law of the Lévy process  $ss$  is fully characterized by the Lévy exponent of  $s(1)$  which is called, by extension the Lévy exponent of the process itself. However, not every Lévy process, and therefore not every white noise, is a valid process in the space of tempered generalized functions since it can increase faster than any polynomial. In Theorem 3, we recall the complete characterization of Lévy exponents related to Lévy processes that are tempered. This result then allows for the proper definition of Lévy white noises in  $\mathcal{S}'(\mathbb{R}^d)$ .

**Theorem 3.** Let  $\psi$  be a continuous function from  $\mathbb{R}$  to  $\mathbb{C}$ . The functional

$$\widehat{\mathcal{P}}(\varphi) = \exp \left( \int_{\mathbb{R}^d} \psi(\varphi(\mathbf{x})) d\mathbf{x} \right) \quad (8)$$

is a valid characteristic functional over  $\mathcal{S}(\mathbb{R}^d)$  if and only if the function  $\psi$  satisfies the two following conditions:

- $\psi$  is a Lévy exponent; and
- the infinitely divisible random variable  $X$  with Lévy exponent  $\psi$  has a finite  $\epsilon$ -moment for some  $\epsilon > 0$ , so that

$$\mathbb{E}[|X|^\epsilon] < \infty. \quad (9)$$

**Definition 5.** A generalized random process  $w$  whose characteristic functional has the form (8) with  $\psi$  satisfying the two conditions of Theorem 3 is called a *Lévy white noise* over  $\mathcal{S}'(\mathbb{R}^d)$ .

Initially, the family of Lévy white noises was introduced over the Schwartz space  $\mathcal{D}'(\mathbb{R}^d)$  of generalized functions, the dual of the space  $\mathcal{D}(\mathbb{R}^d)$  of smooth and compactly supported functions [20, Chapter III]. Theorems 1 and 2 are still valid on  $\mathcal{D}(\mathbb{R}^d)$  which is nuclear. A Lévy white noise on  $\mathcal{D}'(\mathbb{R}^d)$  is not necessarily tempered (an example is given in [15, Section 3.1]). It was shown that the condition (9) for  $w$  being tempered is sufficient in [15, Theorem 3]. More recently, R. Dalang and T. Humeau have proved that this condition is also necessary [9, Theorem 3.13].

A Lévy white noise is stationary in the sense that  $w(\cdot - \mathbf{x}_0)$  and  $w$  have the same law for every  $\mathbf{x}_0 \in \mathbb{R}^d$ . It is moreover independent at every point, meaning that  $\langle w, \varphi_1 \rangle$  and  $\langle w, \varphi_2 \rangle$  are independent whenever  $\varphi_1$  and  $\varphi_2$  have disjoint supports.

As we shall see, one particular subclass of Lévy white noise plays a crucial role as potential scaling limits of general Lévy white noises: the S $\alpha$ S (symmetric- $\alpha$ -stable) white noises.

**Definition 6.** Let  $0 < \alpha \leq 2$ . A Lévy white noise  $w_\alpha$  is a SαS white noise if its characteristic functional has the form

$$\widehat{\mathcal{P}}_{w_\alpha}(\varphi) = \exp(-C\|\varphi\|_\alpha^\alpha) \quad (10)$$

for some  $C > 0$  and every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .

The functional (10) is a valid characteristic functional and correspond to (8) with Lévy exponent  $\psi(\xi) = -C|\xi|^\alpha$ . For every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ , the random variable  $X = \langle w_\alpha, \varphi \rangle$  is SαS with characteristic function  $\widehat{\mathcal{P}}_X(\xi) = \exp(-C\|\varphi\|_\alpha^\alpha |\xi|^\alpha)$ . For  $\alpha = 2$ , one recognizes the Gaussian law. When  $\alpha < 2$ , by contrast, the considered random variables have infinite variances. More information on non-Gaussian SαS random variables and processes can be found on [36].

### 3 Linear SPDE Driven by Lévy White Noises

The general framework to solve linear stochastic differential equations of the form

$$Ls = w, \quad (11)$$

with  $L$  a differential operator and  $w$  a Lévy white noise, is based on the existence of inverse operators with adequate properties [42, Chapter 4]. In this section, we first construct generalized random processes that are solutions of (11) (Section 3.1). Then, we introduce the differential operators that we shall consider thereafter (Section 3.2), and finally define the class of studied processes, called  $\gamma$ -order linear processes (Section 3.3).

#### 3.1 Construction of Linear Processes

Set  $L$  be a continuous and linear operator from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$ . Then, its adjoint is the operator  $L^*$  from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$  defined as

$$\langle L^* \varphi_1, \varphi_2 \rangle = \langle \varphi_1, L \varphi_2 \rangle \quad (12)$$

for every  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ .

**Proposition 1** (Specification of a linear process). *Consider a linear and continuous operator  $L$  from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$  and a Lévy white noise  $w$  on  $\mathcal{S}'(\mathbb{R}^d)$ . Assume the existence of a topological vector space  $\mathcal{X}$  such that*

- *the adjoint  $L^*$  of  $L$  admits a left inverse operator  $T$  that is linear and continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{X}$ ;*
- *the characteristic functional  $\widehat{\mathcal{P}}_w$  of  $w$  can be extended as a continuous and positive-definite functional on  $\mathcal{X}$ .*

Then, there exists a generalized random process  $s$  whose characteristic functional is, for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\widehat{\mathcal{P}}_s(\varphi) = \widehat{\mathcal{P}}_w(T\varphi). \quad (13)$$

Moreover, we have that

$$Ls \stackrel{(d)}{=} w. \quad (14)$$

*Proof.* By considering a more general  $\mathcal{X}$ , this result refines the original theorem that was first presented in [42, Section 3.5.4] albeit with some unnecessary restrictions on  $\mathcal{X}$ . The principle is to check the conditions of the Minlos-Bochner theorem. We detail the proof for the sake of completeness.

We denote  $\widehat{\mathcal{P}}(\varphi) = \widehat{\mathcal{P}}_w(T\varphi)$ . For every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,  $T\{\varphi\} \in \mathcal{X}$  by assumption and the characteristic functional  $\widehat{\mathcal{P}}_w$  is well-defined over  $\mathcal{X}$ . Hence,  $\widehat{\mathcal{P}}$  is well defined over  $\mathcal{S}(\mathbb{R}^d)$ . The operator  $T$  is continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{X}$  and the functional  $\widehat{\mathcal{P}}_w$  is continuous over  $\mathcal{X}$ , implying that  $\widehat{\mathcal{P}}$  is continuous over  $\mathcal{S}(\mathbb{R}^d)$  by composition. It is moreover positive-definite over  $\mathcal{S}(\mathbb{R}^d)$  since  $\widehat{\mathcal{P}}_w$  is positive-definite over  $\mathcal{X}$ . Finally,  $T\{0\} = 0$  by linearity, so that  $\widehat{\mathcal{P}}_s(0) = \widehat{\mathcal{P}}_w(T\{0\}) = \widehat{\mathcal{P}}_w(0) = 1$ . Therefore,  $\widehat{\mathcal{P}}$  is a valid characteristic functional of some generalized random process  $s$  according to Theorem 1.

For the last point, we simply remark that,  $T$  being a left inverse of  $L^*$ ,

$$\widehat{\mathcal{P}}_{Ls}(\varphi) = \mathbb{E} [e^{i\langle Ls, \varphi \rangle}] = \mathbb{E} [e^{i\langle s, L^* \varphi \rangle}] = \widehat{\mathcal{P}}_w(TL^* \varphi) = \widehat{\mathcal{P}}_w(\varphi), \quad (15)$$

which is equivalent to  $Ls \stackrel{(d)}{=} w$ .  $\square$

**Definition 7.** A generalized random process constructed via Proposition 1 is called a *linear process*.

In practice, for given  $L$  and  $w$ , one has to determine an adequate space  $\mathcal{X}$  in order to correctly define the process  $s$ . The choice of  $\mathcal{X}$  is generally driven by the white noise. For instance, we should consider the case  $\mathcal{X} = L^\alpha(\mathbb{R}^d)$  when dealing with  $S\alpha S$  white noises. Then, the main issue becomes the existence of a left inverse with the adequate stability, mapping  $\mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{X}$ .

### 3.2 Differential Operators

This section is dedicated to the description of the class of considered operators. We start with some definitions. For  $u \in \mathcal{S}'(\mathbb{R}^d)$ ,  $\mathbf{x}_0 \in \mathbb{R}^d$ , and  $a > 0$ , we define  $u(\cdot - \mathbf{x}_0)$  as the tempered distribution such that  $\langle u(\cdot - \mathbf{x}_0), \varphi \rangle = \langle u, \varphi(\cdot + \mathbf{x}_0) \rangle$  for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ . Similarly,  $u(\cdot/a)$  is the tempered distribution such that  $\langle u(\cdot/a), \varphi \rangle = \langle u, a^d \varphi(a \cdot) \rangle$ .

**Definition 8.** Consider a linear and continuous operator  $L$  from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$ . We say that  $L$  is  $\gamma$ -homogeneous for some  $\gamma \in \mathbb{R}$  if

$$L\{\varphi(\cdot/a)\} = a^{-\gamma}(L\varphi)(\cdot/a) \quad (16)$$

for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  and  $a > 0$ .

We shall now focus on operators  $L$  that are: (1) linear, (2) continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$ , and (3)  $\gamma$ -homogeneous for some  $\gamma \geq 0$ . Moreover, inspired by Proposition 1, the adjoint operator  $L^*$  should have a left inverse with some stability property. We shall essentially consider two cases, assuming the existence of a left inverse  $T$  as in Proposition 1 for  $\mathcal{X} = \mathcal{R}(\mathbb{R}^d)$ , the space of rapidly decaying measurable functions (see below), or  $\mathcal{X} = L^p(\mathbb{R}^d)$  for some  $p$  such that  $0 < p \leq 2$ . These spaces naturally arise as domains of continuity of the characteristic functional of Lévy white noises.

The space  $\mathcal{R}(\mathbb{R}^d)$  is defined as

$$\mathcal{R}(\mathbb{R}^d) = \{f \text{ measurable, } (1 + |\cdot|)^N f \in L^2(\mathbb{R}^d) \text{ for all } N\}. \quad (17)$$

It is endowed with a natural Fréchet topology, as a projective limit of the Hilbert spaces  $L_n^2(\mathbb{R}^d) = \{f, (1 + |\cdot|)^N f \in L^2(\mathbb{R}^d)\}$ .

Fix a linear, continuous, and  $\gamma$ -homogeneous operator from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$ , with  $\gamma \geq 0$ . We consider two cases.

- **Condition (C1).** The adjoint  $L^*$  admits a  $(-\gamma)$ -homogeneous left inverse that continuously maps  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{R}(\mathbb{R}^d)$ .
- **Condition (C2).** The adjoint  $L^*$  admits a  $(-\gamma)$ -homogeneous left inverse that continuously maps  $\mathcal{S}(\mathbb{R}^d)$  to  $L^p(\mathbb{R}^d)$  for some  $0 < p \leq 2$ .

Note that (C1) is more restrictive than (C2) since  $\mathcal{R}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$  for any  $0 < p \leq 2$ .

### 3.3 Linear Processes of Order $\gamma$

Our goal is to define general classes of random processes solutions of SDEs. In view of Proposition 1, we start by giving some new results on the continuity of the characteristic functionals of Lévy white noises that allow for new compatibility conditions between an operator  $L$  and a Lévy white noise  $w$ .

**Proposition 2.** *Let  $w$  be a Lévy white noise over  $\mathcal{S}'(\mathbb{R}^d)$ . Then, the characteristic functional of  $w$  can be extended as a continuous and positive-definite functional over  $\mathcal{R}(\mathbb{R}^d)$ .*

The proof is given in Appendix A; it relies on our previous work [15].

**Definition 9.** We say that a Lévy exponent  $\psi$  is  $p$ -admissible for  $0 < p \leq 2$  if

$$|\psi(\xi)| \leq C |\xi|^p \quad (18)$$

for some  $C > 0$  and every  $\xi \in \mathbb{R}$ . By extension, a Lévy white noise with a  $p$ -admissible Lévy exponent is said to be  $p$ -admissible itself.

The  $p$ -admissibility allows us to extend the domain of definition of the characteristic functional of the considered white noise.

**Proposition 3.** *Let  $w$  be a Lévy white noise with Lévy exponent  $\psi$ . If  $\psi$  is  $p$ -admissible for some  $0 < p \leq 2$ , then the characteristic functional of  $w$ ,*

$$\widehat{\mathcal{P}}_w(\varphi) = \exp \left( \int_{\mathbb{R}^d} \psi(\varphi(\mathbf{x})) d\mathbf{x} \right), \quad (19)$$

*can be extended as a continuous and positive-definite functional over  $L^p(\mathbb{R}^d)$ .*



The proof of this result, which is very similar to the one of Proposition 2, is given in Appendix A. In [42, Definition 4.4], an alternative definition of the  $p$ -admissibility was introduced. But, since (i) it required an additional bound on the derivative of the Lévy exponent, and (ii) it was used for the construction of sparse processes only for  $1 \leq p \leq 2$ , Proposition 3 is a generalization of [42, Theorem 8.2].

Propositions 2 and 3 allow for new criteria to solve SDEs driven by Lévy white noises.

**Theorem 4.** *Let  $w$  be a Lévy white noise on  $\mathcal{S}'(\mathbb{R}^d)$  with Lévy exponent  $\psi$  and  $L$  be a linear,  $\gamma$ -homogeneous, and continuous operator from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$  for  $\gamma \geq 0$ . We consider two cases.*

- **Condition (C1).** *The adjoint  $L^*$  admits a  $(-\gamma)$ -homogeneous left inverse  $T$  that maps  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{R}(\mathbb{R}^d)$ .*
- **Condition (C2).** *There exists  $0 < p \leq 2$  such that (i) the adjoint  $L^*$  admits a  $(-\gamma)$ -homogeneous left inverse  $T$  that maps  $\mathcal{S}(\mathbb{R}^d)$  to  $L^p(\mathbb{R}^d)$  and (ii)  $\psi$  is  $p$ -admissible.*

*When (C1) or (C2) are satisfied, there exists a generalized random process  $s$  whose characteristic functional is  $\widehat{\mathcal{P}}(\varphi) = \widehat{\mathcal{P}}_w(T\varphi)$ . Moreover,  $s$  is a solution of (11).*

*Proof.* The result follows from the application of Proposition 1 with  $\mathcal{X} = \mathcal{R}(\mathbb{R}^d)$  and  $\mathcal{X} = L^p(\mathbb{R}^d)$ , respectively. The assumptions on  $\widehat{\mathcal{P}}_w$  are satisfied due to Propositions 2 and 3.  $\square$

**Definition 10.** A generalized random process  $s$  constructed with the method of Theorem 4 is called a  $\gamma$ -order linear process. We summarize the situation described in Theorem 4 with the (slightly abusive) notation  $s = L^{-1}w$ .

**Remark.** The Lévy exponent of a S $\alpha$ S white noise is  $\psi(\xi) = -C|\xi|^\alpha$  for some constant  $C > 0$  and thus is  $\alpha$ -admissible. The construction of a process  $s$  such that  $Ls = w_\alpha$  therefore relies on the existence of a left inverse  $T$  of  $L^*$  that maps continuously  $\mathcal{S}(\mathbb{R}^d)$  into  $L^\alpha(\mathbb{R}^d)$ .

## 4 Scaling Limits of $\gamma$ -Order Linear Processes

In this section, we study the statistical behavior of  $\gamma$ -order linear processes at coarse- and fine-scales. We recall that for a generalized random process  $s$  and a nonnegative real number  $a$ , the process  $s(\cdot/a)$  is defined by  $\langle s(\cdot/a), \varphi \rangle = a^d \langle s, \varphi(a \cdot) \rangle$ .

- We zoom out the process when  $a < 1$ . In particular, we consider the limit case  $a \rightarrow 0$  and call it the *coarse-scale behavior* of  $s$ .
- We zoom in the process when  $a > 1$ . Again, we pay attention to the limit case  $a \rightarrow \infty$ , which one call the *fine-scale behavior* of  $s$ .

In general, we shall see that  $s(\cdot/a)$  has no nontrivial limits when  $a \rightarrow 0/\infty$ . However, we shall also encounter situations where  $a^H s(\cdot/a)$  has a stochastic limit for some  $H \in \mathbb{R}$ . When it exists, the coefficient  $H$  is unique and determines the renormalization procedure required to observe the convergence phenomena.

In this section, we first treat the case of SDEs driven by S $\alpha$ S white noises, whose solutions are actually self-similar, so that the scaling limit behaviors are trivial. Second, we introduce the Blumenthal-Gettoor indices of a Lévy exponent, which will appear to be critical for the coarse- and fine-scale behaviors of linear processes. Third, we give sufficient conditions on the Lévy exponent to determine the coarse- and fine-scales behaviors of  $\gamma$ -order linear processes.

#### 4.1 Linear Processes Driven by S $\alpha$ S White Noises

When the white noise is stable, the change of scale has by definition no effect on the noise up to renormalization. Under reasonable assumptions on the operator  $L$ , we extend this fact to solutions of SDEs driven by S $\alpha$ S white noises. This property is referred to as self-similarity.

**Definition 11.** A generalized random process  $s$  is said to be *self-similar of order  $H$*  if

$$a^H s(\cdot/a) \stackrel{(d)}{=} s \quad (20)$$

for all  $a > 0$ . The parameter  $H$  is called the *Hurst exponent* of  $s$ .

The coarse-scale and fine-scale behaviors of a self-similar process are obvious, since the law of the process is not changed by scaling, up to renormalization. The self-similarity property is directly inferred from the characteristic functional of the process. Indeed, since

$$\widehat{\mathcal{P}}_{a^H s(\cdot/a)}(\varphi) = \mathbb{E} \left[ e^{i \langle a^H s(\cdot/a), \varphi \rangle} \right] = \mathbb{E} \left[ e^{i \langle s, a^{d+H} \varphi(a \cdot) \rangle} \right] = \widehat{\mathcal{P}}_s(a^{d+H} \varphi(a \cdot)), \quad (21)$$

we deduce that  $s$  is self-similar of order  $H$  if and only if

$$\widehat{\mathcal{P}}_s(\varphi) = \widehat{\mathcal{P}}_s(a^{d+H} \varphi(a \cdot)) \quad (22)$$

for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  and  $a > 0$ . This equivalence and some other considerations on self-similar processes can be found in [42, Section 7.2].

**Proposition 4.** Let  $\gamma \geq 0$  and  $0 < \alpha \leq 2$ . We assume that  $s = L^{-1} w_\alpha$  is a  $\gamma$ -order linear process driven by a S $\alpha$ S white noise. Then,  $s$  is self-similar with Hurst exponent

$$H = \gamma + d \left( \frac{1}{\alpha} - 1 \right). \quad (23)$$

*Proof.* By definition of a  $\gamma$ -order linear process, there exists a linear operator  $T$ , left inverse of  $L^*$ , that is  $(-\gamma)$ -homogeneous, continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $L^\alpha(\mathbb{R}^d)$ , and such

that  $\widehat{\mathcal{P}}_s(\varphi) = \exp(-C\|\mathsf{T}\varphi\|_\alpha^\alpha)$ . Then, we have

$$\begin{aligned}\widehat{\mathcal{P}}_s(a^{d+H}\varphi(a\cdot)) &= \widehat{\mathcal{P}}_s(a^{\gamma+d/\alpha}\varphi(a\cdot)) \\ &= \exp(-C\|a^{\gamma+d/\alpha}\mathsf{T}\{\varphi(a\cdot)\}\|_\alpha^\alpha) \\ &= \exp(-C\|a^{d/\alpha}\{\mathsf{T}\varphi\}(a\cdot)\|_\alpha^\alpha)\end{aligned}\tag{24}$$

$$\begin{aligned}&= \exp(-C\|\mathsf{T}\varphi\|_\alpha^\alpha) \\ &= \widehat{\mathcal{P}}_s(\varphi),\end{aligned}\tag{25}$$

where we used respectively the  $(-\gamma)$ -homogeneity of  $\mathsf{T}$  and the change of variable  $\mathbf{y} = a\mathbf{x}$  in (24) and (25). According to (22), this implies that  $s$  is self-similar with the Hurst exponent given by (23).  $\square$

## 4.2 Indices of Lévy White Noises

**Definition 12.** The *Blumenthal-Gettoor indices* (or simply, *indices*) of a Lévy exponent  $\psi$  are defined as

$$\beta_0 = \sup \left\{ p \in [0, 2], \quad \limsup_{|\xi| \rightarrow 0} \frac{|\psi(\xi)|}{|\xi|^p} < \infty \right\}, \tag{26}$$

$$\beta_\infty = \inf \left\{ p \in [0, 2], \quad \limsup_{|\xi| \rightarrow \infty} \frac{|\psi(\xi)|}{|\xi|^p} < \infty \right\}. \tag{27}$$

The Blumenthal-Gettoor index  $\beta_\infty$  was initially introduced in [5] to study the local behavior of Lévy processes. The index  $\beta_0$  is the asymptotic counterpart of  $\beta_\infty$  and was considered by B. Pruitt [35]. The index  $\beta_0$  is highly connected to the existence of moments of the infinitely divisible random variable with Lévy exponent  $\psi$ . For instance, for a symmetric finite variance white noise, the index is  $\beta_0 = 2$ . Moreover, one can often characterized  $\beta_0$  with the Lévy measure associated to  $\psi$  (see [11, Section 3]). The fact that a Lévy white noise on  $\mathcal{S}'(\mathbb{R}^d)$  always has a finite moment  $\epsilon > 0$  finite (see Theorem 3) imposes that  $\beta_0 > 0$ . Consequently, we should always consider indices such that  $0 < \beta_0 \leq 2$ . From Definition 12, we can directly deduce Proposition 5.

**Proposition 5.** *Consider a Lévy exponent  $\psi$  with indices  $0 < \beta_0, \beta_\infty \leq 2$ . For  $0 < p \leq 2$ , we have the following relations:*

- If  $\psi(\xi) \underset{\infty}{\sim} -C|\xi|^p$ , then  $\beta_\infty = p$ .
- If  $\psi(\xi) \underset{0}{\sim} -C|\xi|^p$ , then  $\beta_0 = p$ .
- If  $|\psi(\xi)| \leq C|\xi|^p$ , then  $\beta_\infty \leq p \leq \beta_0$ .

The converse of these results are false in general. Note that the last condition of Proposition 5 implies that a  $p$ -admissible Lévy exponent is necessarily associated to indices such that  $\beta_\infty \leq \beta_0$ .

### 4.3 Linear Processes at Coarse- and Fine-Scales

This section is dedicated to the main results of this paper. We analyse the coarse- and fine-scale behavior separately even if the methods of proof are similar, in order to emphasize the different assumptions: the relevant parameter of the underlying white noise is the index  $\beta_0$  at coarse scales and  $\beta_\infty$  at fine scales.

**Theorem 5** (Coarse-scale behavior of  $\gamma$ -order linear processes). *We consider a  $\gamma$ -homogeneous operator  $L$  and a Lévy white noise  $w$  with Lévy exponent  $\psi$  and index  $0 < \beta_0 \leq 2$ . We assume that there exists an operator  $T$  such that  $(T, \psi)$  satisfies one of the following set of conditions.*

- **Condition (C1).**  $T$  is a  $(-\gamma)$ -homogeneous left inverse of  $L^*$ , continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{R}(\mathbb{R}^d)$ ; or
- **Condition (C2).**  $T$  is a  $(-\gamma)$ -homogeneous left inverse of  $L^*$ , continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $L^{\beta_0}(\mathbb{R}^d)$  and  $\psi$  is  $\beta_0$ -admissible.

Let  $s = L^{-1}w$  be the  $\gamma$ -order linear process with characteristic functional  $\widehat{\mathcal{P}}_s(\varphi) = \widehat{\mathcal{P}}_w(T\varphi)$ . Then, if  $\psi(\xi) \sim -C |\xi|^{\beta_0}$  for some constant  $C > 0$ , we have the convergence in law

$$a^{\gamma+d(\frac{1}{\beta_0}-1)} s(\cdot/a) \xrightarrow[a \rightarrow 0]{(d)} s_{L, \beta_0}, \quad (28)$$

where  $Ls_{L, \beta_0} \stackrel{(d)}{=} w_{\beta_0}$  is a  $S\alpha S$  white noise with  $\alpha = \beta_0$ .

*Proof.* First, when (C1) or (C2) holds, Theorem 4 implies that both  $\widehat{\mathcal{P}}_w(T\varphi)$  and  $\widehat{\mathcal{P}}_{w_{\beta_0}}(T\varphi) = \exp(-C\|T\varphi\|_{\beta_0}^{\beta_0})$  are valid characteristic functionals, so that the processes  $s$  and  $s_{L, \beta_0}$  are well defined.

By Theorem 2, we know moreover that the convergence in law (28) is equivalent to the pointwise convergence of the characteristic functionals. Hence, we have to prove that, for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\log \widehat{\mathcal{P}}_{a^{\gamma d(1/\beta_0-1)} s(\cdot/a)}(\varphi) \xrightarrow[a \rightarrow 0]{} -C\|T\varphi\|_{\beta_0}^{\beta_0}. \quad (29)$$

We fix  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ . Then, we have

$$\begin{aligned} \langle a^{\gamma d(1/\beta_0-1)} s(\cdot/a), \varphi \rangle &= \langle w, a^{\gamma+d/\beta_0} \varphi(a \cdot) \rangle \\ &= \langle w, T\{a^{\gamma+d/\beta_0} \varphi(a \cdot)\} \rangle \end{aligned} \quad (30)$$

$$= \langle w, a^{d/\beta_0} \{T\varphi\}(a \cdot) \rangle, \quad (31)$$

where we have used that  $\langle s, \varphi \rangle = \langle w, T\varphi \rangle$  and the  $(-\gamma)$ -homogeneity of  $T$  in (30) and (31), respectively. Therefore, we have

$$\begin{aligned} \log \widehat{\mathcal{P}}_{a^{\gamma d(1/\beta_0-1)} s(\cdot/a)}(\varphi) &= \log \widehat{\mathcal{P}}_w(a^{d/\beta_0} \{T\varphi\}(a \cdot)) \\ &= \int_{\mathbb{R}^d} \psi(a^{d/\beta_0} \{T\varphi\}(a\mathbf{x})) d\mathbf{x} \\ &= \int_{\mathbb{R}^d} (a^{-d} \psi(a^{d/\beta_0} T\varphi(\mathbf{y}))) d\mathbf{y}. \end{aligned} \quad (32)$$

By assumption on  $\psi$ , we have moreover that, for every  $\mathbf{y} \in \mathbb{R}^d$ ,

$$a^{-d}\psi(a^{d/\beta_0}\mathsf{T}\varphi(\mathbf{y})) \xrightarrow{a \rightarrow 0} -C |\mathsf{T}\varphi(\mathbf{y})|^{\beta_0}. \quad (33)$$

We split here the proof in two parts, depending on whether  $\mathsf{T}$  and  $\psi$  follow (C1) or (C2).

- We start with (C2). The  $\beta_0$ -admissibility of  $\psi$  implies that

$$|a^{-d}\psi(a^{d/\beta_0}\mathsf{T}\varphi(\mathbf{y}))| \leq C' |\varphi(\mathbf{y})|^{\beta_0} \quad (34)$$

for some  $C' > 0$  and every  $\mathbf{y} \in \mathbb{R}^d$ . The right term of (34) is integrable by assumption on  $\mathsf{T}$ , so that the Lebesgue dominated-convergence theorem applies and (29) is showed.

- We assume now (C1). In that case, we do not have a full bound on  $\psi$ . However, since  $\psi$  is continuous and behaves like  $(-C |\psi|^{\beta_0})$  at 0, there exists  $C' > 0$  such that  $|\psi(\xi)| \leq C' |\xi|^{\beta_0}$  for every  $|\xi| \leq 1$ . We know moreover that  $\mathsf{T}\varphi$  is bounded. Hence, for all  $a$  such that  $a < \|\mathsf{T}\varphi\|_\infty^{-1}$ , (34) is still valid. Again, we deduce (29) from the Lebesgue dominated convergence theorem.

□

**Theorem 6** (Fine-scale behavior of  $\gamma$ -order linear processes). *Under the same assumptions as in Theorem 5 but replacing  $\beta_0$  by  $\beta_\infty \in (0, 2]$ , we consider  $s = \mathsf{L}^{-1}w$  a  $\gamma$ -order linear process. If the Lévy exponent  $\psi$  of  $w$  satisfies  $\psi(\xi) \underset{\infty}{\sim} -C |\xi|^{\beta_\infty}$  for some constant  $C > 0$ , then we have the convergence in law*

$$a^{\gamma+d(\frac{1}{\beta_\infty}-1)}s(\cdot/a) \xrightarrow{a \rightarrow \infty} s_{\mathsf{L}, \beta_\infty}, \quad (35)$$

where  $\mathsf{L}s_{\mathsf{L}, \beta_\infty} \stackrel{(d)}{=} w_{\beta_\infty}$  is a  $S\alpha S$  white noise with  $\alpha = \beta_\infty$ .

*Proof.* The proof is almost identical to the one of Theorem 5, so that we only develop the parts that differ. If  $\mathsf{T}$  and  $\psi$  satisfy (C2), the proof follows exactly the line of Theorem 5. We should therefore assume that  $\mathsf{T}$  maps continuously  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{R}(\mathbb{R}^d)$ . Restarting from (32) with  $\beta_\infty$  instead of  $\beta_0$ , we split the integral into two parts and get

$$\begin{aligned} \log \widehat{\mathcal{P}}_{a^{\gamma+d(1/\beta_\infty-1)}s(\cdot/a)}(\varphi) &= \int_{\mathbb{R}^d} \mathbb{1}_{|\mathsf{T}\varphi(\mathbf{y})|a^{d/\beta_\infty} \geq 1} a^{-d}\psi(a^{d/\beta_\infty}\mathsf{T}\varphi(\mathbf{y}))d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^d} \mathbb{1}_{|\mathsf{T}\varphi(\mathbf{y})|a^{d/\beta_\infty} < 1} a^{-d}\psi(a^{d/\beta_\infty}\mathsf{T}\varphi(\mathbf{y}))d\mathbf{y} \\ &:= I(a) + J(a). \end{aligned} \quad (36)$$

*Control of  $I(a)$ :* We have, by assumption on  $\psi$ , that

$$\mathbb{1}_{|\mathsf{T}\varphi(\mathbf{y})|a^{d/\beta_\infty} \geq 1} a^{-d}\psi(a^{d/\beta_\infty}\mathsf{T}\varphi(\mathbf{y})) \xrightarrow{a \rightarrow \infty} -C |\mathsf{T}\varphi(\mathbf{y})|^{\beta_\infty}. \quad (37)$$

Moreover, since the continuous function  $\psi$  behaves like  $(-C|\xi|^{\beta_\infty})$  at infinity, there exists a constant  $C'$  such that  $|\psi(\xi)| \leq C'|\xi|^{\beta_\infty}$  for every  $\xi$  with  $|\xi| \geq 1$ . Moreover, the function  $T\varphi$ , which is in  $\mathcal{R}(\mathbb{R}^d)$ , is bounded. Hence, we have, when  $a \geq \|T\varphi\|_\infty^{-1}$ , that

$$\left| \mathbb{1}_{|T\varphi(\mathbf{y})|a^{d/\beta_\infty} \geq 1} a^{-d} \psi(a^{d/\beta_\infty} T\varphi(\mathbf{y})) \right| \leq C' |T\varphi(\mathbf{y})|^{\beta_\infty} \quad (38)$$

for all  $\mathbf{y} \in \mathbb{R}^d$ . The function on the right is integrable, so that the Lebesgue dominated convergence applies and we obtain that  $I(a) \xrightarrow{a \rightarrow \infty} -C\|T\varphi\|_{\beta_\infty}^{\beta_\infty}$ .

*Control of  $J(a)$ :* According to Lemma 3 (see Appendix A), there exists  $C' > 0$  and  $\epsilon > 0$  such that  $|\psi(\xi)| \leq C'(|\xi|^\epsilon + |\xi|^2)$ . Without loss of generality, one can choose  $\epsilon < \beta_\infty$ . Then, for  $|\xi| \leq 1$ , we have  $|\psi(\xi)| \leq 2C'|\xi|^\epsilon$  and, therefore,

$$\left| \int_{\mathbb{R}^d} \mathbb{1}_{|T\varphi(\mathbf{y})|a^{d/\beta_\infty} < 1} a^{-d} \psi(a^{d/\beta_\infty} T\varphi(\mathbf{y})) d\mathbf{y} \right| \leq 2C' a^{d(\epsilon/\beta_\infty - 1)} \|T\varphi\|_\epsilon^\epsilon. \quad (39)$$

Since  $\mathcal{R}(\mathbb{R}^d) \subset L^\epsilon(\mathbb{R}^d)$  and  $\epsilon < \beta_\infty$ , we have  $\|T\varphi\|_\epsilon^\epsilon < \infty$  and  $a^{d(\epsilon/\beta_\infty - 1)} \xrightarrow{a \rightarrow \infty} 0$ , which implies that  $J(a) \xrightarrow{a \rightarrow \infty} 0$ . Finally, we have shown that

$$\log \widehat{\mathcal{P}}_{a^{\gamma+d(1/\beta_\infty-1)}s(\cdot/a)}(\varphi) = I(a) + J(a) \xrightarrow{a \rightarrow \infty} -C|T\varphi(\mathbf{y})|^{\beta_\infty}, \quad (40)$$

as expected.  $\square$

## Remarks

- The renormalization procedures in Theorems 5 and 6 have to be compared with the index  $H = \gamma + d(1/\alpha - 1)$  of a  $\gamma$ -order self-similar process (see Proposition 4). In particular, the  $\gamma$ -order linear processes studied in this section are asymptotically self-similar with index  $\gamma + d(1/\beta_{0/\infty} - 1)$ , where  $\beta_{0/\infty} = \beta_0$  or  $\beta_\infty$ . One can say that the lack of self-similarity of  $s$  is asymptotically removed.
- (C1) has to be understood as the sufficient assumption on the operator  $T$  such that the process  $s$  with characteristic functional  $\widehat{\mathcal{P}}_w(T\varphi)$  is well defined without any additional assumption on the Lévy white noise  $w$ . Therefore, (C1) is restrictive for the operator but not for the noise.
- This is in contrast to (C2). Here, the restriction on  $T$  is minimal since the process  $s_{L, \beta_{0/\infty}}$  should be well defined, and, therefore,  $T$  should at least map  $\mathcal{S}(\mathbb{R}^d)$  into  $L^{\beta_{0/\infty}}(\mathbb{R}^d)$ . It means that (C2) gives sufficient assumptions on the Lévy white noise such that the minimal assumption on  $T$  is also sufficient.
- When the variance of the noise is finite, we have in particular that  $\beta_0 = 2$ . Under the assumptions of Theorem 5, the process  $a^{\gamma-d/2}s(\cdot/a)$  converges to a Gaussian self-similar process. This can be seen as a central limit theorem for  $\gamma$ -order finite-variance linear processes. This finite-variance result was already established in our previous work [16, Theorem 4.2]. Theorem 5 is a generalization for the infinite-variance case. Note, however, that the condition  $\beta_0 = 2$  does not imply that the white noise has a finite variance, so that our new result is also more general for  $\beta_0 = 2$ .

- For important classes of Lévy white noises, the parameter  $\beta_\infty$  vanishes, so that Theorem 6 does not apply. This includes (generalized) Laplace white noises and compound-Poisson white noises (see Section 5.1). In that case, the underlying process does not admit any scaling limit at fine-scales in the sense described in this paper, at least when  $T$  satisfies (C1), as we show in Proposition 6.

**Proposition 6.** *Let  $w$  be a white noise with index  $\beta_\infty = 0$ . Assume that  $L$  is a  $\gamma$ -homogeneous operator and that there exists a  $(-\gamma)$ -homogeneous left inverse  $T$  of  $L^*$  that is continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{R}(\mathbb{R}^d)$ . Let  $s = L^{-1}w$  be the  $\gamma$ -order linear process with characteristic functional  $\widehat{\mathcal{P}}_s(\varphi) = \widehat{\mathcal{P}}_w(T\varphi)$ . Then, for every  $H \in \mathbb{R}$ ,*

$$a^H s(\cdot/a) \xrightarrow[a \rightarrow \infty]{(d)} 0. \quad (41)$$

*Proof.* Due to Theorem 2, we have to show that, for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\log \widehat{\mathcal{P}}_{a^H s(\cdot/a)}(\varphi) \xrightarrow[a \rightarrow \infty]{} 0. \quad (42)$$

Proceeding as in Theorem 5, we easily show that

$$\log \widehat{\mathcal{P}}_{a^H s(\cdot/a)}(\varphi) = \int_{\mathbb{R}^d} a^{-d} \psi(a^{d+H} T\varphi(\mathbf{y})) d\mathbf{y}. \quad (43)$$

According to Lemma 3, there exists  $\epsilon, C' > 0$  such that  $|\psi(\xi)| \leq C' |\xi|^\epsilon$  for  $|\xi| \leq 1$ . Without loss of generality, one can assume that  $\epsilon < \frac{d}{d+|H|}$ . This implies in particular that  $\epsilon(d+H) - d < 0$ . The knowledge that  $\beta_\infty = 0$  is enough to deduce that  $\psi(\xi)$  is also dominated by  $|\xi|$  for  $|\xi| \geq 1$ , so that there exists  $C > 0$  such that

$$|\psi(\xi)| \leq C |\xi|^\epsilon \quad (44)$$

for every  $\xi \in \mathbb{R}$ . Restarting from (43), we obtain that

$$\left| \log \widehat{\mathcal{P}}_{a^H s(\cdot/a)}(\varphi) \right| \leq C \int_{\mathbb{R}^d} a^{\epsilon(d+H)-d} |T\varphi(\mathbf{y})|^\epsilon d\mathbf{y} = C \|T\varphi\|_\epsilon^\epsilon a^{\epsilon(d+H)-d}, \quad (45)$$

which vanishes when  $a \rightarrow \infty$  due to our choice for  $\epsilon$ . This concludes the proof.  $\square$

## 5 Examples of SDEs

### 5.1 Examples of Lévy White Noises

We introduce classical families of Lévy white noises that allow us to illustrate our results.

**From infinitely divisible random variables to Lévy white noises.** Consider a Lévy white noise  $w$  on  $\mathcal{S}'(\mathbb{R}^d)$  and a family of functions  $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$  that converges to  $\mathbb{1}_{[0,1]^d}$  for the topology of  $\mathcal{R}(\mathbb{R}^d)$  (see (17)). Since the characteristic functional of  $w$  is continuous over  $\mathcal{R}(\mathbb{R}^d)$  (Proposition 2), one can show that the sequence  $(\langle w, \varphi_n \rangle)$  is

a Cauchy sequence in  $L^0(\Omega)$ . It therefore converges to some random variable denoted by  $X = \langle w, \mathbb{1}_{[0,1]^d} \rangle$ . This random variable is infinitely divisible, with the characteristic function

$$\widehat{\mathcal{P}}_X(\xi) = \exp \left( \int_{\mathbb{R}^d} \psi(\xi \mathbb{1}_{[0,1]^d}(\mathbf{x})) d\mathbf{x} \right) = \exp(\psi(\xi)). \quad (46)$$

The last equality comes from the fact that  $\psi(0) = 0$ . The law of  $w$  is fully characterized by the law of  $\langle w, \mathbb{1}_{[0,1]^d} \rangle$ . Its probability density function is denoted by  $p_{i.d.}$  to suggest that the law is infinitely divisible.

By convention, the terminology for the random variable  $\langle w, \mathbb{1}_{[0,1]^d} \rangle$  is inherited by the underlying white noise  $w$ . We have already exploited this principle for the definition of S $\alpha$ S white noises, with the particular case of the Gaussian white noise.

**Compound-Poisson White Noises.** A *compound-Poisson white noise* is such that  $\langle w, \mathbb{1}_{[0,1]^d} \rangle$  is a compound-Poisson random variable with characteristic function of the form [42, Section 4.4.2]

$$\widehat{\mathcal{P}}_{\langle w, \mathbb{1}_{[0,1]^d} \rangle}(\xi) = \exp(\lambda(\widehat{\mathcal{P}}_{\text{Jump}}(\xi) - 1)), \quad (47)$$

where  $\lambda > 0$  and  $\widehat{\mathcal{P}}_{\text{Jump}}$  is the characteristic function of a probability law  $\mathcal{P}_{\text{Jump}}$  such that  $\mathcal{P}_{\text{Jump}}\{0\} = 0$ . The notation  $\widehat{\mathcal{P}}_{\text{Jump}}$  is motivated by the fact that the underlying probability law is the common law of the jumps of the compound Poisson white noise [41]. The Lévy exponent of a compound Poisson white noise is bounded, hence its index is  $\beta_\infty = 0$ . This index can take any value in  $(0, 2]$  and is equal to 2 if  $\langle w, \mathbb{1}_{[0,1]^d} \rangle$  is symmetric with a finite variance.

**Generalized Laplace White Noises.** Another interesting infinitely divisible family is given by the generalized-Laplace laws. We follow here the notations of [25]. A *generalized-Laplace white noise* is such that  $\langle w, \mathbb{1}_{[0,1]^d} \rangle$  is a generalized Laplace variable whose characteristic function is given by

$$\widehat{\mathcal{P}}_{\langle w, \mathbb{1}_{[0,1]^d} \rangle}(\xi) = \frac{1}{(1 + \xi^2)^c} = \exp(-c \log(1 + \xi^2)) \quad (48)$$

with  $c > 0$ . When  $c = 1$ , we recognize the Laplace law. The Blumenthal-Gettoor indices of generalized Laplace white noises are  $\beta_\infty = 0$  (since  $\psi$  grows asymptotically slower than any polynomial) and  $\beta_0 = 2$  (symmetric finite-variance white noise), respectively.

**$(\alpha, \beta)$ -Type White Noises.** Finally, we also introduce a new family of white noises to illustrate the richness of the Lévy family. We first need some notation. A *Lévy measure* is a measure  $\nu$  on  $\mathbb{R}$  such that  $\nu\{0\} = 0$  and  $\int_{\mathbb{R}} \inf(1, t^2) \nu(dt) < \infty$ . Then, for  $\nu$  a symmetric Lévy measure, the function  $\psi(\xi) = -\int_{\mathbb{R}} (1 - \cos(\xi t)) \nu(dt)$  is a valid Lévy exponent. This is a particular case of the Lévy-Khintchine decomposition of a Lévy exponent [42, Theorem 4.2]. Then, for  $\alpha, \beta \in (0, 2)$ , we consider the measure

$$\nu_{\alpha, \beta}(dt) = \mathbb{1}_{|t| \leq 1} \frac{dt}{|t|^{\alpha+1}} + \mathbb{1}_{|t| > 1} \frac{dt}{|t|^{\beta+1}}. \quad (49)$$



We easily check that  $\nu_{\alpha,\beta}$  is a symmetric Lévy measure and define therefore the Lévy exponent

$$\psi_{\alpha,\beta}(\xi) = - \int_{\mathbb{R}} (1 - \cos(\xi t)) \nu_{\alpha,\beta}(dt). \quad (50)$$

When  $\alpha = \beta$ , we recover a S $\alpha$ S white noise with Lévy measure  $\nu_{\alpha}(dt) = dt/|t|^{\alpha+1}$ . The Lévy white noise with exponent  $\psi_{\alpha,\beta}$  is called an  $(\alpha, \beta)$ -type white noise. Its interest for our purpose is that it displays all the possible joint behaviors of the Lévy exponent at 0 and infinity, as shown in Proposition 7.

**Proposition 7.** *For  $0 < \alpha, \beta < 2$ , the Lévy exponent  $\psi_{\alpha,\beta}$  satisfies*

$$\psi_{\alpha,\beta}(\xi) \underset{\infty}{\sim} -C_{\infty} |\xi|^{\alpha}, \text{ and} \quad (51)$$

$$\psi_{\alpha,\beta}(\xi) \underset{0}{\sim} -C_0 |\xi|^{\beta}, \quad (52)$$

with  $C_0, C_{\infty} > 0$  some constants.

*Proof.* We have

$$\psi_{\alpha,\beta}(\xi) = - \int_{|t| \leq 1} (1 - \cos(\xi t)) \frac{dt}{|t|^{\alpha+1}} - \int_{|t| > 1} (1 - \cos(\xi t)) \frac{dt}{|t|^{\beta+1}} := \psi_1(\xi) + \psi_2(\xi). \quad (53)$$

Then, by the change of variable  $x = \xi t$ , we have that

$$\psi_1(\xi) = - \left( \int_{|x| \leq |\xi|} (1 - \cos x) \frac{dx}{|x|^{\alpha+1}} \right) |\xi|^{\alpha} \underset{\infty}{\sim} - \left( \int_{\mathbb{R}} (1 - \cos x) \frac{dx}{|x|^{\alpha+1}} \right) |\xi|^{\alpha} \quad (54)$$

while  $|\psi_2(\xi)| \leq \int_{|t| > 1} 2 \frac{dt}{|t|^{\beta+1}} = o(|\xi|^{\alpha})$ , implying (51) with  $C_{\infty} = \int_{|x| \leq |\xi|} (1 - \cos x) \frac{dx}{|x|^{\alpha+1}}$ .

Similarly, we have that

$$\psi_2(\xi) \underset{0}{\sim} - \left( \int_{\mathbb{R}} (1 - \cos x) \frac{dx}{|x|^{\beta+1}} \right) |\xi|^{\beta} \quad (55)$$

while  $|\psi_1(\xi)| \leq \frac{1}{2} \left( \int_{|t| \leq 1} \frac{dt}{|t|^{\alpha-1}} \right) |\xi|^2 = o(|\xi|^{\beta})$ , where we have used that  $|1 - \cos(\xi t)| \leq \frac{\xi^2 t^2}{2}$ . This implies (52) with  $C_0 = \int_{\mathbb{R}} (1 - \cos x) \frac{dx}{|x|^{\beta+1}}$ .  $\square$

Due to Proposition 7, the  $\gamma$ -order linear processes based on an  $(\alpha, \beta)$ -type white noise have the interesting following property: While failing to be self-similar, they offer a transition between a self-similarity of order  $H_{\text{fine}} = \gamma + d(1/\alpha - 1)$  at fine scale to a self-similarity of order  $H_{\text{coarse}} = \gamma + d(1/\beta - 1)$  at coarse scale. This can be of interest for modeling purposes.

**Summary.** By studying the behavior of the Lévy exponent at 0 and  $\infty$  (as we did for  $\psi_{\alpha,\beta}$ ), one easily obtains the indices of the Lévy white noises of Table 1.

Table 1: Some Lévy white noises with their Blumenthal-Gettoor indices.

| White noise                                | Parameter                                | $\psi(\xi)$   | $\beta_0$ | $\beta_\infty$ |
|--|--|---|-----------|----------------|
| Gaussian                                   | $\sigma^2 > 0$                           | $-\sigma^2 \xi^2/2$                                     | 2         | 2              |
| Non-Gaussian SαS                           | $\alpha \in (0, 2)$                      | $- \xi ^\alpha$   | $\alpha$  | $\alpha$       |
| Generalized Laplace                        | $c > 0$                                  | $-c \log(1 + \xi^2)$                                    | 2         | 0              |
| Symmetric finite-variance compound Poisson | $\lambda > 0, \mathcal{P}_{\text{Jump}}$ | $\lambda(\widehat{\mathcal{P}}_{\text{Jump}}(\xi) - 1)$ | 2         | 0              |
| Compound Poisson with SαS jumps            | $\lambda > 0, \alpha \in (0, 2)$         | $\lambda(e^{- \xi ^\alpha} - 1)$                        | $\alpha$  | 0              |
| $(\alpha, \beta)$ -type                    | $\alpha, \beta \in (0, 2)$               | $\psi_{\alpha, \beta}(\xi)$ (see (50))                  | $\alpha$  | $\beta$        |

## 5.2 Lévy Processes and Sheets

The canonical basis of  $\mathbb{R}^d$  is  $(\mathbf{e}_k)_{k=1\dots d}$ . We denote by  $D_k$  the partial derivative along the direction  $\mathbf{e}_k$ . Then, a *Lévy sheet* in dimension  $d$  is a solution of

$$Ls = D_1 \cdots D_d s = w \quad (56)$$

with  $w$  a  $d$ -dimensional Lévy white noise [10]. When  $d = 1$ , one recognizes the family of Lévy processes that corresponds to the differential equation  $Ds = w$  in dimension  $d = 1$ .

The linear operator  $L = D_1 \cdots D_d$  is continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}(\mathbb{R}^d)$  and  $d$ -homogeneous. Its adjoint  $L^* = (-1)^d D_1 \cdots D_d$  admits the natural  $(-d)$ -homogeneous (left and right) inverse defined by

$$(L^*)^{-1} \varphi(\mathbf{x}) = \int_{(-\infty, x_1) \times \cdots \times (-\infty, x_d)} \varphi(\mathbf{t}) d\mathbf{t} \quad (57)$$

for  $\mathbf{x} = (x_1, \dots, x_d)$  and  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ . Unfortunately,  $(L^*)^{-1}$  is unstable in the sense that it does not map  $\mathcal{S}(\mathbb{R}^d)$  in any  $L^p(\mathbb{R}^d)$  space,  $0 < p \leq 2$  (and, *a fortiori*, not in  $\mathcal{R}(\mathbb{R}^d)$ ). We can however correct  $(L^*)^{-1}$  to transform it into a stable *left* inverse. For this, we define  $T$  as the adjoint of the operator

$$T^* \varphi(\mathbf{x}) = \int_{(0, x_1) \times \cdots \times (0, x_d)} \varphi(\mathbf{t}) d\mathbf{t}. \quad (58)$$

The operator  $T$  is  $(-d)$ -homogeneous and continuous from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{R}(\mathbb{R}^d)$  [15, Section 4.2]. We satisfy therefore the Condition (C1) of Theorem 4 and define  $s = (D_1 \cdots D_d)^{-1} w$  with characteristic functional  $\widehat{\mathcal{P}}_s(\varphi) = \widehat{\mathcal{P}}_w(T\varphi)$  for any white noise  $w$ . Note that this way of defining  $s$  can be interpreted in terms of boundary conditions—it imposes that  $s(x_1, \dots, x_d) = 0$  if at least one of the  $x_k$  is 0. In particular, in dimension  $d = 1$ , it imposes that  $s(0) = 0$ .

Applying the results of Section 4.3, we directly deduce Proposition 8.

**Proposition 8.** *Consider  $w$  a Lévy white noise with indices  $0 < \beta_0, \beta_\infty \leq 2$ , and  $s = (D_1 \cdots D_d)^{-1} w$  as above. Then,*

- if  $\psi(\xi) \sim -C |\xi|^{\beta_0}$  for some  $C > 0$ , then  $a^{d/\beta_0} s(\cdot/a) \xrightarrow{a \rightarrow 0} s_{D_1 \cdots D_d, \beta_0}$ ;
- if  $\psi(\xi) \sim -C |\xi|^{\beta_\infty}$  for some  $C > 0$ , then  $a^{d/\beta_\infty} s(\cdot/a) \xrightarrow{a \rightarrow \infty} s_{D_1 \cdots D_d, \beta_\infty}$ .

Here,  $s_{D_1, \dots, D_d, \alpha} = (D_1 \cdots D_d)^{-1} w_\alpha$ , where  $w_\alpha$  is a SαS white noise.

### 5.3 Fractional Lévy Processes and Fields

In dimension  $d$ , we consider the stochastic differential equation

$$Ls = (-\Delta)^{\gamma/2} s = w, \quad (59)$$

where  $(-\Delta)^{\gamma/2}$  is the fractional Laplacian whose Fourier multiplier is  $\|\omega\|^\gamma$  with  $\gamma \geq 0$  and  $\gamma/2 \notin \mathbb{N}$ . The fractional Laplacian is self-adjoint and  $\gamma$ -homogeneous. For  $p \geq 1$  such that  $(\gamma + d/p - 1) \notin \mathbb{N}$ ,  $(-\Delta)^{\gamma/2}$  admits a (unique)  $(-\gamma)$ -homogeneous left inverse  $T_{\gamma,p}$  that continuously map  $\mathcal{S}(\mathbb{R}^d)$  into  $L^p(\mathbb{R}^d)$  [39, Theorem 3.7]. For such  $p$ , if the Lévy white noise is  $p$ -admissible, we satisfy Condition (C2) of Theorem 4 and define  $s = ((-\Delta)^{\gamma/2})^{-1}w$  with characteristic functional  $\widehat{\mathcal{P}}_s(\varphi) = \widehat{\mathcal{P}}_w(T_{\gamma,p}\varphi)$ . The process  $s$  is called a fractional Lévy process (a fractional Lévy field when  $d \geq 2$ ).

Again, the direct application of the results of Section 4.3 yields Proposition 9.

**Proposition 9.** *For  $\gamma, p$  satisfying the conditions above, consider  $w$  a  $p$ -admissible Lévy white noise with indices  $0 < \beta_0, \beta_\infty \leq 2$ , and  $s = ((-\Delta)^{\gamma/2})^{-1}w$  as above. Then,*

- *if  $\psi(\xi) \underset{0}{\sim} -C|\xi|^{\beta_0}$  for some  $C > 0$ , then  $a^{\gamma+d(1/\beta_0-1)}s(\cdot/a) \xrightarrow{a \rightarrow 0} s_{(-\Delta)^{\gamma/2}, \beta_0}$ ;*
- *if  $\psi(\xi) \underset{\infty}{\sim} -C|\xi|^{\beta_\infty}$  for some  $C > 0$ , then  $a^{\gamma+d(1/\beta_\infty-1)}s(\cdot/a) \xrightarrow{a \rightarrow \infty} s_{(-\Delta)^{\gamma/2}, \beta_\infty}$ .*

Here,  $s_{(-\Delta)^{\gamma/2}, \alpha} = (-\Delta)^{-\gamma/2}w_\alpha$ , where  $w_\alpha$  is a  $S\alpha S$  white noise.

In dimension  $d = 1$ , one could consider the case of the fractional derivative  $L = D^\gamma$  in a very similar fashion. This includes in particular the fractional Brownian motions [29] and its Lévy-driven generalizations. The construction of stable inverses of the adjoint is the subject of [42, Section 5.5.1].

## A Continuity Results on the Characteristic Functional of Lévy White Noises

The aim of this appendix is to prove Proposition 2 and Proposition 3. We start with some preliminary lemmas.

**Lemma 1.** *If  $\widehat{\mathcal{P}}$  is a positive-definite functional over a topological vector space  $E$  with  $\widehat{\mathcal{P}}(0) = 1$ , then*

$$\left| \widehat{\mathcal{P}}(\varphi) \right| \leq 1, \quad (60)$$

$$\left| \widehat{\mathcal{P}}(\varphi_2) - \widehat{\mathcal{P}}(\varphi_1) \right| \leq 2 \left( 1 - \Re \{ \widehat{\mathcal{P}}(\varphi_2 - \varphi_1) \} \right) \quad (61)$$

for every  $\varphi, \varphi_1, \varphi_2 \in E$ .

This result is classical and shows in particular that a positive-definite functional that is continuous around 0 is uniformly continuous; see for instance [18, Section II.5.1] or [43, Section IV.1.2, Proposition 1.1] for a more complete proof.

**Lemma 2.** *Let  $w$  be a Lévy white noise with Lévy exponent  $\psi$ . Then, there exists a constant  $C > 0$  such that for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,*

$$\left| 1 - \widehat{\mathcal{P}}_w(\varphi) \right| \leq C \int_{\mathbb{R}^d} |\psi(\varphi(\mathbf{x}))| d\mathbf{x}. \quad (62)$$

*Proof.* According to [17, Lemma 1], there exists a constant  $C$  such that  $|1 - e^z| \leq C(1 - e^{-|z|})$  for all complex numbers  $z$  such that  $\Re\{z\} \leq 0$ . Since  $\Re\{\psi\} \leq 0$ , we can apply this result to deduce that

$$\begin{aligned} \left| 1 - \widehat{\mathcal{P}}_w(\varphi) \right| &= \left| 1 - e^{\int \psi(\varphi)} \right| \\ &\leq C \left| 1 - e^{-\left| \int \psi(\varphi) \right|} \right| \\ &\leq C \left| 1 - e^{-\int |\psi(\varphi)|} \right| \\ &\leq C \int |\psi(\varphi)|, \end{aligned} \quad (63)$$

where the last inequality comes from the relation  $(1 - e^{-x}) \leq x$ , valid for  $x \geq 0$  and where the two previous ones come from the fact that  $x \mapsto 1 - e^{-x}$  is increasing.  $\square$

**Lemma 3.** *Let  $w$  be a Lévy white noise on  $\mathcal{S}'(\mathbb{R}^d)$  and  $\psi$  its Lévy exponent. Then, there exists  $\epsilon > 0$  such that*

$$|\psi(\xi)| \leq C(|\xi|^\epsilon + |\xi|^2) \quad (64)$$

for some  $C > 0$  and every  $\xi \in \mathbb{R}$ .

*Proof.* Let  $X$  be the infinitely divisible random variable with Lévy exponent  $\psi$ . According to Theorem 3, there exists  $\epsilon$  such that  $\mathbb{E}[|X|^\epsilon] < \infty$ . Without loss of generality, we can assume that  $\epsilon < 1$ . Then, we are in the condition of [15, Corollary 1], from which we deduce the existence of a constant  $C > 0$  such that (64) holds. Note that the mentioned result deals with a Lévy exponent without a Gaussian part, but this Gaussian part is of the form  $\psi_{\text{Gauss}}(\xi) = \mu\xi + \sigma^2\xi^2/2$  and is itself dominated by  $(|\xi|^\epsilon + |\xi|^2)$  since  $\epsilon < 1$ . Therefore, the result is still valid for a general  $\psi$ .  $\square$

We are now ready to prove the results.

*Proof of Proposition 2.* Combining Lemmas 1 and 2, we deduce that, for every  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} \left| \widehat{\mathcal{P}}_w(\varphi_2) - \widehat{\mathcal{P}}_w(\varphi_1) \right| &\leq 2 \left( 1 - \Re\{ \widehat{\mathcal{P}}_w(\varphi_2 - \varphi_1) \} \right) \\ &= 2 \Re \left\{ 1 - \widehat{\mathcal{P}}_w(\varphi_2 - \varphi_1) \right\} \\ &\leq 2 \left| 1 - \widehat{\mathcal{P}}_w(\varphi_2 - \varphi_1) \right| \\ &\leq C' \int |\psi(\varphi_1 - \varphi_2)|. \end{aligned} \quad (65)$$

From Lemma 3, we then deduce the existence of  $\epsilon > 0$  and  $C'' > 0$  such that

$$\left| \widehat{\mathcal{P}}_w(\varphi_2) - \widehat{\mathcal{P}}_w(\varphi_1) \right| \leq C''(\|\varphi\|_\epsilon^\epsilon + \|\varphi\|_2^2). \quad (66)$$

*Extension of  $\widehat{\mathcal{P}}_w$ .* Let  $\varphi \in \mathcal{R}(\mathbb{R}^d)$ . The space  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $\mathcal{R}(\mathbb{R}^d)$ . By density, there exists a sequence of functions  $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$  such that  $\varphi_n \xrightarrow{n \rightarrow \infty} \varphi$  in  $\mathcal{R}(\mathbb{R}^d)$ . For every  $n, m$ , we have, due to (65),

$$\left| \widehat{\mathcal{P}}_w(\varphi_n) - \widehat{\mathcal{P}}_w(\varphi_m) \right| \leq C''(\|\varphi_n - \varphi_m\|_\epsilon^\epsilon + \|\varphi_n - \varphi_m\|_2^2). \quad (67)$$

Moreover,  $\mathcal{R}(\mathbb{R}^d)$  is continuously embedded into all the spaces  $L^p(\mathbb{R}^d)$ ,  $p > 0$ , so that the right term in the inequation (67) vanishes when  $n, m \rightarrow \infty$ . Hence,  $(\widehat{\mathcal{P}}_w(\varphi_n))$  is a Cauchy sequence in  $\mathbb{C}$ . We call  $\widehat{\mathcal{P}}_w(\varphi)$  its limit. We see easily that this limit does not depend on the sequence  $(\varphi_n)$ , since  $\left| \widehat{\mathcal{P}}_w(\varphi_n) - \widehat{\mathcal{P}}_w(\tilde{\varphi}_n) \right| \leq C''(\|\varphi_n - \tilde{\varphi}_n\|_\epsilon^\epsilon + \|\varphi_n - \tilde{\varphi}_n\|_2^2) \xrightarrow{n} 0$  for two sequences approximating  $\varphi$  in  $\mathcal{R}(\mathbb{R}^d)$ .

*Continuity of  $\widehat{\mathcal{P}}_w$ .* Let  $\varphi, \tilde{\varphi} \in \mathcal{R}(\mathbb{R}^d)$  and  $(\varphi_n), (\tilde{\varphi}_n)$  be two sequences that approximate the functions  $\varphi$  and  $\tilde{\varphi}$  in  $\mathcal{R}(\mathbb{R}^d)$ , respectively. We fix  $\delta > 0$ . Then, we have

$$\begin{aligned} \left| \widehat{\mathcal{P}}_w(f) - \widehat{\mathcal{P}}_w(g) \right| &\leq \left| \widehat{\mathcal{P}}_w(f) - \widehat{\mathcal{P}}_w(\varphi_n) \right| + \left| \widehat{\mathcal{P}}_w(\varphi_n) - \widehat{\mathcal{P}}_w(\tilde{\varphi}_n) \right| + \left| \widehat{\mathcal{P}}_w(\tilde{\varphi}_n) - \widehat{\mathcal{P}}_w(g) \right| \\ &\leq C''(\|\varphi_n - \tilde{\varphi}_n\|_\epsilon^\epsilon + \|\varphi_n - \tilde{\varphi}_n\|_2^2) \\ &\quad + \left| \widehat{\mathcal{P}}_w(\varphi) - \widehat{\mathcal{P}}_w(\varphi_n) \right| + \left| \widehat{\mathcal{P}}_w(\tilde{\varphi}_n) - \widehat{\mathcal{P}}_w(\tilde{\varphi}) \right|. \end{aligned} \quad (68)$$

For  $n$  big enough, we have that  $\|\varphi_n - \tilde{\varphi}_n\|_\epsilon^\epsilon + \|\varphi_n - \tilde{\varphi}_n\|_2^2 \leq \|\varphi - \tilde{\varphi}\|_\epsilon^\epsilon + \|\varphi - \tilde{\varphi}\|_2^2 + \delta$ ,  $\left| \widehat{\mathcal{P}}_w(\varphi) - \widehat{\mathcal{P}}_w(\varphi_n) \right| \leq \delta$  and  $\left| \widehat{\mathcal{P}}_w(\tilde{\varphi}_n) - \widehat{\mathcal{P}}_w(\tilde{\varphi}) \right| \leq \delta$ . Hence,

$$\left| \widehat{\mathcal{P}}_w(\varphi) - \widehat{\mathcal{P}}_w(\tilde{\varphi}) \right| \leq C''(\|\varphi - \tilde{\varphi}\|_\epsilon^\epsilon + \|\varphi - \tilde{\varphi}\|_2^2) + (C'' + 2)\delta. \quad (69)$$

This is true for  $\delta$  arbitrarily small, implying that, for every  $\varphi, \tilde{\varphi} \in \mathcal{R}(\mathbb{R}^d)$ ,

$$\left| \widehat{\mathcal{P}}_w(\varphi) - \widehat{\mathcal{P}}_w(\tilde{\varphi}) \right| \leq C''(\|\varphi - \tilde{\varphi}\|_\epsilon^\epsilon + \|\varphi - \tilde{\varphi}\|_2^2) \quad (70)$$

and that the functional  $\widehat{\mathcal{P}}_w$  is continuous over  $\mathcal{R}(\mathbb{R}^d)$ .

*Positive-definiteness of  $\widehat{\mathcal{P}}_w$ .* We show that  $\widehat{\mathcal{P}}_w$  is positive-definite using a density argument. We fix  $N \geq 1$ ,  $\varphi_1, \dots, \varphi_N \in \mathcal{R}(\mathbb{R}^d)$ ,  $a_1, \dots, a_N \in \mathbb{C}$ . For  $k = 1, \dots, N$ , we choose a sequence  $(\varphi_{k,n})$  that converges to  $\varphi_k$  in  $\mathcal{R}(\mathbb{R}^d)$ . Then, we have, using the continuity of  $\widehat{\mathcal{P}}_w$ , that

$$\begin{aligned} \sum_{k,l=1}^N a_k a_l^* \widehat{\mathcal{P}}_w(\varphi_k - \varphi_l) &= \sum_{k,l=1}^N a_k a_l^* \lim_{n \rightarrow \infty} \widehat{\mathcal{P}}_w(\varphi_{k,n} - \varphi_{l,n}) \\ &= \lim_{n \rightarrow \infty} \sum_{k,l=1}^N a_k a_l^* \widehat{\mathcal{P}}_w(\varphi_{k,n} - \varphi_{l,n}) \\ &\geq 0, \end{aligned} \quad (71)$$

since  $\widehat{\mathcal{P}}_w$  is positive-definite over  $\mathcal{S}(\mathbb{R}^d)$ . Therefore,  $\widehat{\mathcal{P}}_w$  is positive-definite over  $\mathcal{R}(\mathbb{R}^d)$ .  $\square$

*Proof of Proposition 3.* The proof is very similar to the one of Proposition 2, so that we should only briefly describe the steps that differ. The  $p$ -admissibility of  $\psi$  implies that, for every  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\int |\psi(\varphi)| \leq C \|\varphi\|_p^p. \quad (72)$$

This implies, using (65), that, for every  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} \left| \widehat{\mathcal{P}}_w(\varphi_2) - \widehat{\mathcal{P}}_w(\varphi_1) \right| &\leq C' \int |\psi(\varphi_1 - \varphi_2)| \\ &\leq C'' \|\varphi_1 - \varphi_2\|_p^p. \end{aligned} \quad (73)$$

*Extension of  $\widehat{\mathcal{P}}_w$ .* We extend  $\widehat{\mathcal{P}}_w$  by approximating  $\varphi \in L^p(\mathbb{R}^d)$  by functions in  $\mathcal{S}(\mathbb{R}^d)$ . Note indeed that the space  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ , as is well known for  $p \geq 1$  and still true for  $0 < p < 1$ .

*Continuity of  $\widehat{\mathcal{P}}_w$ .* As for Proposition 2, we show that the relation

$$\left| \widehat{\mathcal{P}}_w(\varphi_2) - \widehat{\mathcal{P}}_w(\varphi_1) \right| \leq C''' \|\varphi_1 - \varphi_2\|_p^p \quad (74)$$

holds for  $\varphi_1, \varphi_2 \in L^p(\mathbb{R}^d)$ .

*Positive-definiteness of  $\widehat{\mathcal{P}}_w$ .* The positive-definiteness over  $L^p(\mathbb{R}^d)$  follows from a density argument.  $\square$

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